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1. For constant $c, \alpha_1, \ldots, \alpha_n$ and an n-dimensional Brownian Motion $B_t = (B_1(t), \ldots, B_n(t))$, let

$$X_t = \exp(ct + \sum_{j=1}^{n} \alpha_j B_j(t)) \tag{1}$$

Prove that

$$dX_t = (c + \frac{1}{2}\sum_{j=1}^{n} \alpha_j^2) X_t + X_t(\sum_{j=1}^{n} \alpha_j dB_j(t)) \tag{2}$$

Proof. Pick $u(t, B_1, \ldots, B_n) = \exp(ct + \sum_{j=1}^{n} \alpha_j B_j(t))$. The partial derivatives are

$$u_0 = \frac{\partial u}{\partial t} = c\exp(ct + \sum_{j=1}^{n} \alpha_j B_j(t)) = cX_t \tag{3}$$

$$u_j = \frac{\partial u}{\partial B_j(t)} = \alpha_j \exp(ct + \sum_{j=1}^{n} \alpha_j B_j(t)) = \alpha_j X_t \tag{4}$$

$$u_{i,j} = \frac{\partial^2 u}{\partial B_i(t) \partial B_j(t)} = \delta_{ij}\alpha_j^2 \exp(ct + \sum_{j=1}^{n} \alpha_j B_j(t)) = \delta_{ij}\alpha_j^2 X_t \tag{5}$$

where δ_{ij} is the *Kronecker delta*. Apply Itô's lemma to get:

$$dX_t = u_0 dt + \sum_{j=1}^{n} u_j dB_j(t) + \frac{1}{2}\sum_{j=1}^{n} u_{jj} dt \tag{6}$$

$$= cX_t dt + \sum_{j=1}^{n} \alpha_j dB_j(t) X_t + \frac{1}{2} X_t \sum_{j=1}^{n} \alpha_j^2 dt \tag{7}$$

$$= (c + \frac{1}{2}\sum_{j=1}^{n} \alpha_j^2) X_t dt + X_t \sum_{j=1}^{n} \alpha_j dB_j(t) \tag{8}$$

which completes the proof. □

2. Solve the mean-reverting Ornstein-Uhlenbeck equation:

$$dX_t = (m - X_t)dt + \sigma dB_t \tag{9}$$

where m and σ are real constants. For a constant initial condition $\hat{x}$ calculate the mean and variance of X_t.

Solution. Differentiate the function $f(t, X_t) = X_t e^t$

$$df = X_t e^t dt + e^t dX_t \quad (10)$$
$$= m e^t dt + \sigma e^t dB_t \quad (11)$$

Integrate

$$X_t e^t = \hat{x} + m \int_0^t e^s ds + \sigma \int_0^t e^s dB_s \quad (12)$$
$$X_t = \hat{x} e^{-t} + m(1 - e^{-t}) + \sigma \int_0^t e^{s-t} dB_s \quad (13)$$

To calculate the mean, note that $E[B_s] = 0$, which eliminates the contribution of the integral.

$$E[X_t] = E[\hat{x} e^{-t} + m(1 - e^{-t}) + \int_0^t e^{s-t} dB_s] \quad (14)$$
$$= \hat{x} e^{-t} + m(1 - e^{-t}) + E[\int_0^t e^{s-t} dB_s] \quad (15)$$
$$= \hat{x} e^{-t} + m(1 - e^{-t}) \quad (16)$$

Calculate the variance

$$E[X_t^2] = E[\sigma^2 \int_0^t e^{u-t} du \int_0^t e^{v-t} dv] \text{ by Itô isometry} \quad (17)$$
$$= \frac{\sigma^2}{2} e^{-2t}(e^{2t} - 1) \quad (18)$$
$$= \frac{\sigma^2}{2}(1 - e^{-2t}) \quad (19)$$

3. Let $x > 0$ be a constant and let

$$X_t = (x^{\frac{1}{3}} + \frac{1}{3} B_t)^3; \quad t \geq 0 \quad (20)$$

Prove that

$$dX_t = \frac{1}{3} X_t^{\frac{1}{3}} dt + X_t^{\frac{2}{3}} dB_t; \quad X_0 = x. \quad (21)$$

Proof. Define $u(t, B_t) = X_t = (x^{\frac{1}{3}} + \frac{1}{3} B_t)^3$. Find the partial derivatives

$$u_0 = \frac{\partial u}{\partial t} = 0 \quad (22)$$
$$u_1 = \frac{\partial u}{\partial B_t} = (x^{\frac{1}{3}} + \frac{1}{3} B_t)^2 = X_t^{\frac{2}{3}} \quad (23)$$
$$u_{11} = \frac{\partial^2 u}{\partial B_t \partial B_t} = \frac{2}{3}(x^{\frac{1}{3}} + \frac{1}{3} B_t)^{\frac{3}{3}} = \frac{2}{3} X_t^{\frac{1}{3}} \quad (24)$$

Now apply Itô's lemma

$$\begin{aligned} dX_t &= u_0 dt + u_1 dB_t + u_{11} dt & (25)\\ &= \frac{1}{2}\frac{2}{3} X_t^{\frac{2}{3}} dt + X_t^{\frac{1}{3}} dB_t & (26)\\ &= \frac{1}{3} X_t^{\frac{2}{3}} dt + X_t^{\frac{1}{3}} dB_t & (27) \end{aligned}$$

which completes the proof. □

4. Consider the stochastic differential equation

$$dX_t = f(X_t)dt + e(X_t)dB_t;\ \ X_0 = x \tag{28}$$

Assume that e does not vanish and let g be a C^2 function, satisfying

$$f(x)g'(x) + \frac{1}{2}e^2(x)g''(x) = 0. \tag{29}$$

Let (a, b) be an open interval such that $x \in (a, b)$. Define

$$\tau = \inf\{t > 0;\ \ X_t \notin (a, b)\} \tag{30}$$

and

$$p = \Pr[X_\tau = b]. \tag{31}$$

Prove that

$$p = \frac{g(x) - g(a)}{g(b) - g(a)} \tag{32}$$

Proof. Itô's formula states that if g satisfies (29), $g(X_t)$ is a martingale starting at the initial point x. By above definition (30), X_t exits the interval (a, b) at either a or b. Use the martingale property

$$\begin{aligned} E[g(X_t)] &= \lim_{t\to\infty} E[g(X_t)\cdot 1_{\{t\le\tau\}}] & (33)\\ &= g(x) & (34)\\ E[\tau] &< \infty & (35) \end{aligned}$$

Follow the steps from class (CMG) and apply the relation

$$1 - p = \Pr[X_\tau = a] \tag{36}$$

to get

$$g(a)(1 - p) + g(b)p = g(x). \tag{37}$$

Solve for p:

$$p = \frac{g(x) - g(a)}{g(b) - g(a)} \tag{38}$$

□

Find a formula for p when

$$X_t = x + ct + \sigma B_t \tag{39}$$

Solution. Solve the ODE (29) for $f(x) = c$ and $e(x) = \sigma$. The solution is found using (1) reduction of order and then (2) integrating factors (using the assumption that $\sigma \neq 0$). The solution is $g(x) = \alpha e^{-\frac{2cx}{\sigma^2}}$. Plug this expression into (38), apply condition that $\Pr[X_\tau = b] = 1$ at $x = b$ and get:

$$p(x) = \frac{1 - e^{-\frac{2c(x-a)}{\sigma^2}}}{1 - e^{-\frac{2c(b-a)}{\sigma^2}}} \tag{40}$$